数理解析研究所講究録 1363 巻 2004 年 10-20

On a subclass of analytic functions involving certain fractional calculus operators

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Abstract

Let $\mathcal{A}$ be the class of normalized analytic functions in the open unit disk $\mathbb{U}$. We consider a subclass $\mathcal{A}(\alpha,\beta,\gamma)$ of $\mathcal{A}$ which is defined by using certain fractional calculus operators. The main object of this paper is to investigate subordination theorems, argument theorems and the Fekete-Szegö problem of maximizing $|a_3-\mu a_2^2|$ for functions belonging to the class $\mathcal{A}(\alpha,\beta,\gamma)$, where μ is real. We also obtain certain class-preserving integral operators for the class $\mathcal{A}(\alpha,\beta,\gamma)$.

1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of functions $f(z)$ normalized by

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U}=\{z : z\in\mathbb{C} \text{ and } |z|<1\}$. Also let $\mathcal{S}$, $\mathcal{S}^*(\gamma)$ and $\mathcal{K}(\gamma)$ denote, respectively, the subclasses of $\mathcal{A}$ consisting of functions which are univalent, starlike of order γ and convex of order γ in $\mathbb{U}$ (see, e.g., [15]). In particular, the classes $\mathcal{S}^*(0)=\mathcal{S}^*$ and $\mathcal{K}(0)=\mathcal{K}$ are the familiar classes of starlike and convex functions in $\mathbb{U}$, respectively.

Given two functions $f(z)$ and $g(z)$, which are analytic in $\mathbb{U}$ with $f(0)=g(0)$, $f(z)$ is said to be subordinate to $g(z)$ if there exists an analytic function $w(z)$ on $\mathbb{U}$ such that $w(0)=0$, $|w(z)|<1$ and $f(z)=g(w(z))$ for $z\in\mathbb{U}$. We denote this subordination by

$$f(z)\prec g(z)\quad \text{in } \mathbb{U}.$$

Note that if $g(z)$ is univalent in $\mathbb{U}$, then $f(z)\prec g(z)$ if and only if $f(0)=g(0)$ and $f(\mathbb{U})\subset g(\mathbb{U})$.

2000 *Mathematics Subject Classification.* Primary 30C45, 26A33.

Key words and phrases. Analytic functions, starlike functions, convex functions, fractional derivative, fractional integral.

Several essentially equivalent definitions of fractional calculus (that is, fractional integral and fractional derivative) have been studied in the literature (*cf.*, *e.g.*, [3], [11] and [12, p.28 *et seq.*]). We state the following definitions due to Owa [8] which have been used rather frequently in the theory of analytic functions (see also [10] and [14]).

Definition 1. The fractional integral of order λ $(\lambda > 0)$ is defined, for a function $f(z)$, by

$$\mathcal{D}_z^{-\lambda} f(z) := \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta, \tag{1.2}$$

and the fractional derivative of order λ $(0 \leq \lambda < 1)$ by

$$\mathcal{D}_z^{\lambda} f(z) := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta, \tag{1.3}$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ involved in (1.2) (and that of $(z-\zeta)^{-\lambda}$ in (1.3)) is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

Definition 2. Under the hypotheses of Definition 1, the fractional derivative of order $n+\lambda$ $(0 \leq \lambda < 1; n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$ is defined by

$$\mathcal{D}_z^{n+\lambda} f(z) := \frac{d^n}{dz^n} \mathcal{D}_z^{\lambda} f(z). \tag{1.4}$$

With the aid of the above definitions, Owa and Srivastava [10] defined the fractional calculus operator $\mathcal{J}_z^{\lambda}$ $(\lambda \in \mathbb{R}; \lambda \neq 2, 3, 4, \cdots)$ by

$$\mathcal{J}_z^{\lambda} f(z) = \Gamma(2-\lambda) z^{\lambda} \mathcal{D}_z^{\lambda} f(z) \tag{1.5}$$

for functions (1.1) belonging to the class $\mathcal{A}$.

Recently, Choi *et al.* [2] investigated the subclass $\mathcal{A}(\alpha, \beta, \gamma)$ of $\mathcal{A}$ for $\alpha < 2$, $\beta < 2$ and $\gamma < 1$, which was defined by

$$\mathcal{A}(\alpha, \beta, \gamma) := \left\{ f \in \mathcal{A} : \mathrm{Re} \left(\frac{\mathcal{J}_z^{\alpha} f(z)}{\mathcal{J}_z^{\beta} f(z)} \right) > \gamma \text{ in } \mathbb{U} \right\}. \tag{1.6}$$

We note that $\mathcal{A}(1, 0, \gamma) = \mathcal{S}^*(\gamma)$ and $\mathcal{A}(\lambda+1, 0, \gamma) = \mathcal{S}^*(\gamma, \lambda)$ $(\lambda < 1; 0 \leq \gamma < 1)$ which was studied by Owa and Shen [9]. Recently, Srivastava *et al.* [13] proved inclusion and subordination properties of the class $\mathcal{A}(\lambda+1, \lambda, (\rho-\lambda)/(1-\lambda)) = \mathcal{S}_{\lambda}(\rho)$ $(0 \leq \lambda < 1; 0 \leq \rho < 1)$.

In this paper, we investigate subordination theorems, argument theorems and the upper bound of the quantity $a_3 - \mu a_2^2$ for functions belonging to the class $\mathcal{A}(\alpha, \beta, \gamma)$, where μ is real. We also consider certain class-preserving integral operators for the class $\mathcal{A}(\alpha, \beta, \gamma)$.

2. Preliminary results

In order to prove our results, we need the following lammas.

Lemma 1. (Choi *et al.* [2]) *Let* $\lambda < 1$ *and* $f(z) \in \mathcal{A}$. *Then*

$$z\left(\mathcal{J}_z^{\lambda} f(z)\right)' = (1-\lambda)\mathcal{J}_z^{\lambda+1} f(z) + \lambda \mathcal{J}_z^{\lambda} f(z) \qquad (z \in \mathbb{U}), \tag{2.1}$$

where the operator $\mathcal{J}_z^{\lambda}$ *is given by* (1.5).

Lemma 2. (Hallenbeck and Ruscheweyh [4]) *Let* $g(z)$ *be convex univalent in* $\mathbb{U}$ *with* $g(0) = 1$. *If* $\mathrm{Re}(\eta) > 0$ *and* $f(z)$ *is analytic in* $\mathcal{D}$ *with* $f(z) \prec g(z)$, *then*

$$\frac{1}{z^{\eta}} \int_0^z f(t) t^{\eta-1} dt \prec \frac{1}{z^{\eta}} \int_0^z g(t) t^{\eta-1} dt. \tag{2.2}$$

Lemma 3. (Jack [5]) *Let* $w(z)$ *be analytic in* $\mathbb{U}$ *with* $w(0) = 0$. *Then if* $|w(z)|$ *attains its maximum value on the circle* $|z| = r$ $(r < 1)$ *at a point* z_0, *we can write*

$$z_0 w'(z_0) = k w(z_0),$$

where k *is real and* $k \geq 1$.

Lemma 4. (Ma and Minda [7]) *Let* $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ *be analytic in* $\mathbb{U}$ *with* $\mathrm{Re}\, p(z) > 0$ $(z \in \mathbb{U})$. *Then*

$$|c_2 - \nu c_1^2| \leq \begin{cases} -4\nu + 2 & if\ \nu \leq 0 \\ 2 & if\ 0 \leq \nu \leq 1 \\ 4\nu - 2 & if\ \nu \geq 1. \end{cases} \tag{2.3}$$

3. Subordination and argument theorems

First, by using Lemma 2, we prove

Theorem 1. *Let* $\alpha < 2$, $\beta < 2$ *and* $\gamma < 1$. *If* $f(z) \in \mathcal{A}(\alpha, \beta, \gamma)$, *then*

$$\frac{1}{z} \int_0^z \left(\frac{\mathcal{J}_z^{\alpha} f(t)}{\mathcal{J}_z^{\beta} f(t)} \right) dt \prec 2\gamma - 1 - \frac{2(1-\gamma)}{z} \log(1-z). \tag{3.1}$$

Proof. Let $f(z) \in \mathcal{A}(\alpha, \beta, \gamma)$ and set

$$g(z) = \frac{1 + (1-2\gamma)z}{1-z} \qquad (z \in \mathbb{U})$$

which maps the unit disk $\mathbb{U}$ onto the half domain such that Re $(w) > \gamma$. Then, from the definition of the class $\mathcal{A}(\alpha, \beta, \gamma)$ we have

$$\frac{\mathcal{J}_z^\alpha f(z)}{\mathcal{J}_z^\beta f(z)} \prec g(z) = \frac{1+(1-2\gamma)z}{1-z}. \tag{3.2}$$

Furthermore, the function $g(z)$ is convex univalent in $\mathbb{U}$ with $g(0) = 1$. Hence, by applying Lemma 2 with $\eta = 1$, we observe that

$$\frac{1}{z}\int_0^z \left(\frac{\mathcal{J}_z^\alpha f(t)}{\mathcal{J}_z^\beta f(t)}\right) dt \prec \frac{1}{z}\int_0^z \frac{1+(1-2\gamma)t}{1-t} dt$$

which yields (3.1).

Remark 1. If $\alpha = \lambda + 1$ and $\beta = 0$ in Theorem 1, then it would immediately yield the result of Owa and Shen [9, Theorem 2.1].

Corollary 1. *Let* $\lambda < 1$ *and* $\gamma < 1$. *If* $f(z) \in \mathcal{A}(\lambda+1, \lambda, \gamma)$, *then*

$$\frac{1}{z}\int_0^z \left(\frac{t\left(\mathcal{J}_z^\lambda f(t)\right)'}{\mathcal{J}_z^\lambda f(t)}\right) dt \prec 2\lambda - 1 + 2(1-\lambda)\left(\gamma - \frac{1-\gamma}{z}\log(1-z)\right). \tag{3.3}$$

Proof. Let $f(z) \in \mathcal{A}(\lambda+1, \lambda, \gamma)$. Then, by using Lemma 1, it is easily verified that

$$\mathrm{Re}\left(\frac{z\left(\mathcal{J}_z^\lambda f(z)\right)'}{\mathcal{J}_z^\lambda f(z)}\right) > (1-\lambda)\gamma + \lambda. \tag{3.4}$$

Hence, by using the same techniques as in the proof of Theorem 1 with

$$g(z) = \frac{1+(2(1-\gamma)(1-\lambda)-1)\,z}{1-z}, \tag{3.5}$$

we conclude that

$$\frac{1}{z}\int_0^z \left(\frac{t\left(\mathcal{J}_z^\lambda f(t)\right)'}{\mathcal{J}_z^\lambda f(t)}\right) dt \prec \frac{1}{z}\int_0^z \frac{1+(2(1-\gamma)(1-\lambda)-1)\,t}{1-t} dt$$

which evidently implies (3.3).

Putting $\gamma = 0$ in Theorem 1, we obtain

Corollary 2. *Let* $\alpha < 2$ *and* $\beta < 2$. *If* $f(z) \in \mathcal{A}(\alpha, \beta, 0)$, *then*

$$\frac{1}{z}\int_0^z \left(\frac{\mathcal{J}_z^\alpha f(t)}{\mathcal{J}_z^\beta f(t)}\right) dt \prec -1 - \frac{2}{z}\log(1-z).$$

Next, we derive the arguments for functions belonging to the class $\mathcal{A}(\alpha,\beta,\gamma)$.

Theorem 2. *Let* $\alpha < 2$, $\beta < 2$ *and* $\gamma < 1$. *If* $f(z) \in \mathcal{A}(\alpha,\beta,\gamma)$, *then*

$$\left|\arg\left(z^{\alpha-\beta}\frac{\mathcal{D}_z^{\alpha}f(z)}{\mathcal{D}_z^{\beta}f(z)}\right)\right| \leq \sin^{-1}\left(\frac{2(1-\gamma)|z|}{1+(1-2\gamma)|z|^2}\right) \qquad (z \in \mathbb{U}) \tag{3.6}$$

and

$$\frac{1-(1-2\gamma)|z|}{1+|z|} \leq \left|\frac{\mathcal{J}_z^{\alpha}f(z)}{\mathcal{J}_z^{\beta}f(z)}\right| \leq \frac{1+(1-2\gamma)|z|}{1-|z|} \qquad (z \in \mathbb{U}).$$

Proof. Since $f(z) \in \mathcal{A}(\alpha,\beta,\gamma)$, in view of (3.2), we can write

$$\frac{\mathcal{J}_z^{\alpha}f(z)}{\mathcal{J}_z^{\beta}f(z)} = \frac{1+(1-2\gamma)w(z)}{1-w(z)}, \tag{3.7}$$

where $w(z)$ is analytic in $\mathbb{U}$ with $w(0)=0$ and $|w(z)|<1$. We now consider the function

$$h(z) = \frac{1+Aw(z)}{1+Bw(z)} \qquad (-1 \leq B < A; z \in \mathbb{U}).$$

It is well known that $h(z)$, for $-1 \leq B \leq 1$, is the conformal map of the disk $|w(z)| < |z|$ onto the disk

$$\left|h(z) - \frac{1-AB|z|^2}{1-B^2|z|^2}\right| \leq \frac{(A-B)|z|}{1-B^2|z|^2}. \tag{3.8}$$

By virtue of (3.7) and (3.8), we have

$$\left|\frac{\mathcal{J}_z^{\alpha}f(z)}{\mathcal{J}_z^{\beta}f(z)} - \frac{1+(1-2\gamma)|z|^2}{1-|z|^2}\right| \leq \frac{2(1-\gamma)|z|}{1-|z|^2}, \tag{3.9}$$

which immediately yields the assertion (3.6).

Moreover, it follows from (3.9) that

$$\frac{1-(1-2\gamma)|z|}{1+|z|} \leq \left|\frac{\mathcal{J}_z^{\alpha}f(z)}{\mathcal{J}_z^{\beta}f(z)}\right| \leq \frac{1+(1-2\gamma)|z|}{1-|z|}.$$

This completes the proof of Theorem 2.

Corollary 3. *Let* $\lambda < 1$ *and* $\gamma < 1$. *If* $f(z) \in \mathcal{A}(\lambda+1,\lambda,\gamma)$, *then*

$$\left|\arg\left(\frac{z\left(\mathcal{D}_z^{\lambda}f(z)\right)'}{\mathcal{D}_z^{\lambda}f(z)}\right)\right| \leq \sin^{-1}\left(\frac{2(1-\gamma)|z|}{1+(1-2\gamma)|z|^2}\right) \qquad (z \in \mathbb{U})$$

and

$$\frac{(1-\lambda)(1-(1-2\gamma)|z|)}{1+|z|} \leq \left|\frac{z\left(\mathcal{D}_z^{\lambda}f(z)\right)'}{\mathcal{D}_z^{\lambda}f(z)}\right| \leq \frac{(1-\lambda)(1+(1-2\gamma)|z|)}{1-|z|} \qquad (z \in \mathbb{U}).$$

Proof. In view of (3.4) and (3.5), we set

$$\frac{z\left(\mathcal{J}_z^{\lambda} f(z)\right)'}{\mathcal{J}_z^{\lambda} f(z)} = \frac{1 + (2(1-\gamma)(1-\lambda) - 1)\, w(z)}{1 - w(z)} \qquad (z \in U).$$

Here $w(z)$ is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$. Then, by using same argument of Theorem 2, we can easily verify Corollary 3, and so we omit it.

4. Coefficient bound and class-preserving integral operators

We begin by applying Lemma 4 to prove

Theorem 3. *Let $\beta < \alpha < 2$, $\gamma < 1$ and $\mu \in \mathbb{R}$. If $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{A}(\alpha, \beta, \gamma)$, then*

$$\left|a_3 - \mu a_2^2\right|$$

$$\leq \begin{cases} \left(\dfrac{2(\alpha-\beta+2(1-\gamma)(2-\alpha))}{\alpha-\beta} - \dfrac{6(1-\gamma)(2-\alpha)(2-\beta)(5-\alpha-\beta)}{(\alpha-\beta)(3-\alpha)(3-\beta)}\mu\right) K \\ \qquad \text{if } \mu \leq \dfrac{2(3-\alpha)(3-\beta)}{3(2-\beta)(5-\alpha-\beta)} \\ 2K \qquad \text{if } \dfrac{2(3-\alpha)(3-\beta)}{3(2-\beta)(5-\alpha-\beta)} \leq \mu \leq \dfrac{2(3-\alpha)(3-\beta)(2-\beta-\gamma(2-\alpha))}{3(2-\alpha)(2-\beta)(1-\gamma)(5-\alpha-\beta)} \\ \left(\dfrac{6(1-\gamma)(2-\alpha)(2-\beta)(5-\alpha-\beta)}{(\alpha-\beta)(3-\alpha)(3-\beta)}\mu - \dfrac{2(\alpha-\beta+2(1-\gamma)(2-\alpha))}{\alpha-\beta}\right) K \\ \qquad \text{if } \dfrac{2(3-\alpha)(3-\beta)(2-\beta-\gamma(2-\alpha))}{3(2-\alpha)(2-\beta)(1-\gamma)(5-\alpha-\beta)} \leq \mu, \end{cases}$$

where

$$K = \frac{(1-\gamma)(2-\alpha)(2-\beta)(3-\alpha)(3-\beta)}{6(\alpha-\beta)(5-\alpha-\beta)}. \tag{4.1}$$

Proof. If we set

$$p(z) = \frac{\dfrac{\mathcal{J}_z^{\alpha} f(z)}{\mathcal{J}_z^{\beta} f(z)} - \gamma}{1-\gamma} = 1 + c_1 z + c_2 z^2 + \cdots \qquad (f \in \mathcal{A}), \tag{4.2}$$

then $p(z)$ is analytic with $p(0) = 1$ and has a positive real part in $\mathbb{U}$. In view of (4.2), a simple calculation shows

$$a_2 = \frac{(1-\gamma)(2-\alpha)(2-\beta)}{2(\alpha-\beta)} c_1 \tag{4.3}$$

and

$$a_3 = K\left(c_2 + \frac{(1-\gamma)(2-\alpha)}{\alpha-\beta}c_1^2\right), \tag{4.4}$$

where K is given by (4.1). Therefore, using (4.3) and (4.4), we see that

$$|a_3 - \mu a_2^2| = K\,|c_2 - \nu c_1^2|,$$

where

$$\nu = \frac{3(1-\gamma)(2-\alpha)(2-\beta)(5-\alpha-\beta)}{2(\alpha-\beta)(3-\alpha)(3-\beta)}\mu - \frac{(1-\gamma)(2-\alpha)}{\alpha-\beta}.$$

Hence, by applying Lemma 4, we obtain the desired result. We omit further details.

Setting $\alpha = \beta + 1$ in Theorem 3, we have

Corollary 4. *Let $\beta < 1$, $\gamma < 1$ and $\mu \in \mathbb{R}$. If $f(z) \in \mathcal{A}(\beta+1,\beta,\gamma)$, then*

$|a_3 - \mu a_2^2|$

$$\leq \begin{cases} (1-\gamma)(1-\beta)(2-\beta)\left[(3-\beta)\left(\frac{1}{2}-\frac{1}{3}(\beta+\gamma(1-\beta))\right) - (1-\gamma)(1-\beta)(2-\beta)\mu\right] \\ \quad if\ \ 3(2-\beta)\mu \leq 3-\beta \\ \frac{(1-\gamma)(1-\beta)(2-\beta)(3-\beta)}{6} \qquad if\ \ \frac{3-\beta}{3(2-\beta)} \leq \mu \leq \frac{(3-\beta)(1+(1-\gamma)(1-\beta))}{3(1-\gamma)(1-\beta)(2-\beta)} \\ (1-\gamma)(1-\beta)(2-\beta)\left[(1-\gamma)(1-\beta)(2-\beta)\mu - (3-\beta)\left(\frac{1}{2}-\frac{1}{3}(\beta+\gamma(1-\beta))\right)\right] \\ \quad if\ \ (3-\beta)(1+(1-\gamma)(1-\beta)) \leq 3(1-\gamma)(1-\beta)(2-\beta)\mu. \end{cases}$$

Remark 2. If $\gamma = (\rho-\beta)/(1-\beta)$ $(0 \leq \beta < 1; 0 \leq \rho < 1)$ in Corollary 4, then it would immediately yields the result of Srivastava *et al.* [13, Theorem 4].

Next, we consider the generalized Bernardi-Libera-Livingston integral operator I_c $(c > -1)$ defined by (cf. [1], [6] and [15])

$$I_c(f)(z) := \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \qquad (f \in \mathcal{A}; c > -1). \tag{4.5}$$

It follows from (4.5) that

$$\begin{aligned} I_c(f)(z) &= \frac{c+1}{z^c}\int_0^z \left(t^c + \sum_{n=2}^{\infty} a_n t^{n+c-1}\right) dt \\ &= z + \sum_{n=2}^{\infty} \frac{c+1}{c+n} a_n z^n. \end{aligned} \tag{4.6}$$

Theorem 4. *Let $\lambda < 1$, $\gamma < 1$ and $c \geq -\lambda - (1-\lambda)\gamma$. Suppose that $f(z) \in \mathcal{A}(\lambda+1, \lambda, \gamma_0)$, where*

$$\gamma_0 \equiv \gamma_0(c,\gamma,\lambda) = \begin{cases} \gamma - \dfrac{(1-\gamma)(1-\lambda)}{2(c+\lambda+\gamma(1-\lambda))} & \text{if } (1-\lambda)(1-2\gamma)-\lambda \leq c \\[2ex] \gamma - \dfrac{c+\lambda+\gamma(1-\lambda)}{2(1-\gamma)(1-\lambda)} & \text{if } (1-\lambda)(1-2\gamma)-\lambda \geq c. \end{cases} \tag{4.7}$$

Then $I_c(f)(z) \in \mathcal{A}(\lambda+1, \lambda, \gamma)$.

Proof. Making use of (1.5) and (4.6), we obtain

$$\begin{aligned} z\left(\mathcal{J}_z^\lambda I_c(f)(z)\right)' &= z + \sum_{n=2}^{\infty} \frac{n(c+1)}{c+n} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+\lambda-1)} a_n z^n \\ &= z + \sum_{n=2}^{\infty} \left(c+1-\frac{c(c+1)}{c+n}\right) \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+\lambda-1)} a_n z^n \\ &= (c+1)\mathcal{J}_z^\lambda f(z) - c\mathcal{J}_z^\lambda I_c(f)(z). \end{aligned} \tag{4.8}$$

Define the function $w(z)$ by

$$\frac{z\left(\mathcal{J}_z^\lambda I_c(f)(z)\right)'}{\mathcal{J}_z^\lambda I_c(f)(z)} = \frac{1+(2(1-\gamma)(1-\lambda)-1)\,w(z)}{1-w(z)} \qquad (z \in \mathbb{U}). \tag{4.9}$$

Then $w(z)$ is analytic in $\mathbb{U}$ with $w(0)=0$ and $w(z) \neq -1$. Hence, by applying the method of the aforementioned of [2, Theorem 4] with (4.8) and (4.9), we can easily prove Theorem 4, and so we omit the details.

Finally, we state and prove

Theorem 5. *Let $c \geq 0$, $\alpha < 2$, $\beta < 1$ and $\gamma < 1$. Suppose that $f(z) \in \mathcal{A}(\alpha,\beta,\gamma) \bigcap \mathcal{A}(\beta+1,\beta,\gamma_1)$, where*

$$\gamma_1 \equiv \gamma_1(c,\beta) = \begin{cases} \dfrac{\beta(1-2c)-1}{2c(1-\beta)} & \text{if } 1 \leq c \\[2ex] \dfrac{\beta(c-2)-c}{2(1-\beta)} & \text{if } 0 \leq c \leq 1. \end{cases} \tag{4.10}$$

Then $I_c(f)(z) \in \mathcal{A}(\alpha,\beta,\gamma)$.

Proof. This proof is much akin to that of [9, Theorem 6.1], so we shall omit some details here. If we define the function $w(z)$ by

$$\frac{\mathcal{J}_z^\alpha I_c(f)(z)}{\mathcal{J}_z^\beta I_c(f)(z)} = \frac{1+(1-2\gamma)w(z)}{1-w(z)} \qquad (\gamma < 1; z \in \mathbb{U}), \tag{4.11}$$

then $w(z)$ is analytic in $\mathbb{U}$ with $w(0)=0$ and $w(z)\neq -1$. We need to show that $|w(z)|<1$ for all $z\in\mathbb{U}$. Thus, by using similar way as in the proof of [9, Theorem 6.1] with Lemma 3, and putting $w(z_0)=e^{i\theta}$, we observe that

$$\mathrm{Re}\left(\frac{\mathcal{J}_z^{\alpha}f(z_0)}{\mathcal{J}_z^{\beta}f(z_0)}\right)=\gamma-\frac{k(1-\gamma)}{1-\cos\theta}\mathrm{Re}\left(\frac{1}{\frac{z_0(\mathcal{J}_z^{\beta}I_c(f)(z_0))'}{\mathcal{J}_z^{\beta}I_c(f)(z_0)}+c}\right)\qquad (k\geq 1). \tag{4.12}$$

Since $f(z)\in\mathcal{A}(\beta+1,\beta,\gamma_1)$, in view of Theorem 4, we have

$$I_c(f)(z)\in\mathcal{A}\left(\beta+1,\beta,-\frac{\beta}{1-\beta}\right). \tag{4.13}$$

Therefore, it follows from (2.1) and (4.13) that

$$\begin{aligned}&\mathrm{Re}\left(\frac{1}{\frac{z_0(\mathcal{J}_z^{\beta}I_c(f)(z_0))'}{\mathcal{J}_z^{\beta}I_c(f)(z_0)}+c}\right)\\ &=\frac{(1-\beta)\mathrm{Re}\left(\frac{\mathcal{J}_z^{\beta+1}I_c(f)(z_0)}{\mathcal{J}_z^{\beta}I_c(f)(z_0)}\right)+\beta+c}{\left[\mathrm{Re}\left(\frac{z_0(\mathcal{J}_z^{\beta}I_c(f)(z_0))'}{\mathcal{J}_z^{\beta}I_c(f)(z_0)}+c\right)\right]^2+\left[\mathrm{Im}\left(\frac{z_0(\mathcal{J}_z^{\beta}I_c(f)(z_0))'}{\mathcal{J}_z^{\beta}I_c(f)(z_0)}+c\right)\right]^2}>0.\end{aligned} \tag{4.14}$$

Consequently, we obtain that

$$\mathrm{Re}\left(\frac{\mathcal{J}_z^{\alpha}f(z)}{\mathcal{J}_z^{\beta}f(z)}\right)\leq\gamma$$

which contradicts the hypothesis $f(z)\in\mathcal{A}(\alpha,\beta,\gamma)$. Hence $|w(z)|<1$ for all $z\in\mathbb{U}$, and by (4.11), we have the desired result.

Remark 3. Taking $\alpha=\lambda+1$ and $\beta=0$ in Theorem 5, we see that

$$f(z)\in\mathcal{S}^*(\gamma,\lambda)\bigcap\mathcal{A}(1,0,\gamma_1(c,0))\quad\text{implies}\quad I_c(f)(z)\in\mathcal{S}^*(\gamma,\lambda),$$

where $\gamma_1(c,0)$ is given by (4.10). Since $\gamma_1(c,0)\leq 0$,

$$\mathcal{S}^*=\mathcal{A}(1,0,0)\subset\mathcal{A}(1,0,\gamma_1(c,0)).$$

Hence Theorem 5 provides a improvement of the result due to Owa and Shen [9, Theorem 6.1].

Corollary 5. *Let $c\geq 0$, $\beta<1$ and $\gamma_1\leq\gamma<1$, where γ_1 is given by* (4.10). *If $f(z)\in\mathcal{A}(\beta+1,\beta,\gamma)$, then $I_c(f)(z)\in\mathcal{A}(\beta+1,\beta,\gamma)$.*

Proof. Since $\gamma_1\leq\gamma<1$, in view of (1.6), we obtain

$$\mathcal{A}(\beta+1,\beta,\gamma)\bigcap\mathcal{A}(\beta+1,\beta,\gamma_1)=\mathcal{A}(\beta+1,\beta,\gamma).$$

Hence, by virtue of Theorem 5, we conclude that

$$f(z) \in \mathcal{A}(\beta+1,\beta,\gamma) \Longrightarrow I_c(f)(z) \in \mathcal{A}(\beta+1,\beta,\gamma),$$

which completes the proof of Corollary 5.

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